SOME RECENT RESULTS IN RAMSEY THEORY

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Abstract. We review and comment on a number of results in Ramsey theory obtained recently by the author in collaboration with V. Kanellopoulos, N. Karagiannis and K. Tyros. Among them are density versions of the classical pigeonhole principles of Halpern–Läuchli and Carlson–Simpson.

We shall comment on recent progress concerning one fundamental problem in Ramsey theory. It originates from an insightful conjecture of Erdős and Turán [20] and, in full generality, asks to determine which pigeonhole principles admit a density version. There have been numerous dramatic developments in this direction—see, e.g., [4, 26, 27, 39, 46] and the references therein—which go well beyond the scope of the present review. We will thus be forced to neglect a vast amount of remarkable current research. We shall focus, instead, on three basic pigeonhole principles which are particularly appealing due to their widespread utility and unifying power.

1. The coloring versions

1.1. The first pigeonhole principle relevant to our discussion is the Hales–Jewett theorem [28]. To state it we need to introduce some pieces of notation and some terminology. For every integer $k \geqslant 2$ let $[k]^{<\mathbb{N}}$ be the set of all finite sequences having values in $[k] := \{1, \dots, k\}$. The elements of $[k]^{<\mathbb{N}}$ are referred to as *words over* k, or simply *words* if k is understood. If $n \in \mathbb{N}$, then $[k]^n$ stands for the set of words of length n. We fix a letter v that we regard as a variable. A *variable word over* k is a finite sequence having values in $[k] \cup \{v\}$ where the letter v appears at least once. If w is a variable word and $a \in [k]$, then $w(a)$ is the word obtained by substituting all appearances of the letter v in w by a. A *combinatorial line* of $[k]^n$ is a set of the form $\{w(a) : a \in [k]\}$ where w is a variable word over k of length n.

Hales–Jewett theorem. *For every $k, r \in \mathbb{N}$ with $k \geqslant 2$ and $r \geqslant 1$ there exists a positive integer N with the following property. If $n \geqslant N$, then for every r-coloring of $[k]^n$ there exists a combinatorial line of $[k]^n$ which is monochromatic. The least positive integer with this property will be denoted by* $\mathrm{HJ}(k, r)$.

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The Hales–Jewett theorem is often regarded as an abstract version of the van der Waerden theorem [50] and is considered to be one of the cornerstones of contemporary Ramsey theory. The best known upper bounds for the numbers $\mathrm{HJ}(k,r)$ are primitive recursive and are due to Shelah [40].

1.2. The second pigeonhole principle relevant to our discussion is the Halpern–Läuchli theorem [29], a rather deep result that concerns partitions of finite products of infinite trees. It was discovered in 1966, three years after the discovery of the Hales–Jewett theorem, as a result needed for the construction of a model of set theory in which the boolean prime ideal theorem is true but not the full axiom of choice. Since then, it has been the main tool for the development of Ramsey theory for trees, a rich area of combinatorics with significant applications, most notably in the geometry of Banach spaces; see, for instance, [6, 8, 9, 25, 31, 32, 42, 49] and [1, 2, 10, 11, 43, 48] for applications.

The Halpern–Läuchli theorem has several equivalent forms. We will state the "strong subtree version" which is the most important one from a combinatorial perspective.

Halpern–Läuchli theorem. *For every finite tuple $(T_1,\dots,T_d)$ of uniquely rooted and finitely branching trees without maximal nodes and every finite coloring of the level product*

$$\bigcup_{n\in\mathbb{N}} T_1(n)\times\cdots\times T_d(n) \tag{1}$$

of $(T_1,\dots,T_d)$, there exist strong subtrees $(S_1,\dots,S_d)$ of $(T_1,\dots,T_d)$ of infinite height and with a common level set such that their level product is monochromatic.

We recall that a subtree S of a tree $(T,<)$ is said to be *strong* if: (a) S is uniquely rooted and balanced[1], (b) every level of S is a subset of some level of T, and (c) for every non-maximal node $s\in S$ and every immediate successor t of s in T there exists a unique immediate successor s' of s in S with $t\leqslant s'$. The last condition is the most important one and expresses a basic combinatorial requirement, namely that a strong subtree of T must respect the "tree structure" of T. The *level set* of a strong subtree S of a tree T is the set of levels of T containing a node of S.

Although the notion of a strong subtree was isolated in the 1960s, it was highlighted with the work of Milliken in [34, 35] who used the Halpern–Läuchli theorem to show that the family of strong subtrees of a uniquely rooted and finitely branching tree is partition regular.

1.3. The Hales–Jewett theorem and the Halpern–Läuchli theorem are pigeonhole principles of quite different nature. Nevertheless, they do admit a common extension which is due to Carlson and Simpson [9]. To state it we recall that a *left*

[1]A tree S is *balanced* if all maximal chains of S have the same cardinality.

variable word over k is a variable word over k whose leftmost letter is the variable v. The concatenation of two words x and y over k is denoted by $x^\frown y$.

Carlson–Simpson theorem. *For every integer* $k \geqslant 2$ *and every finite coloring of the set of all words over* k *there exist a word* c *over* k *and a sequence* (w_n) *of left variable words over* k *such that the set*

$$\{c\} \cup \big\{c^\frown w_0(a_0)^\frown \dots ^\frown w_n(a_n) : n \in \mathbb{N} \text{ and } a_0, \dots, a_n \in [k]\big\} \tag{2}$$

is monochromatic.

The Carlson–Simpson theorem belongs to the circle of results that refine the Hales–Jewett theorem by providing information on the structure of the wildcard[2] set of the monochromatic variable word; see, e.g., [5, 30, 33, 41, 51]. This extra information (namely, that the sequence (w_n) consists of left variable words) can then be used to derive the Halpern–Läuchli theorem when the trees $T_1, \dots, T_d$ are homogeneous[3], a special case which is sufficient for all known combinatorial applications of the Halpern–Läuchli theorem (see [37]).

2. The density versions

It is a remarkably fruitful phenomenon that several pigeonhole principles have a density version. These density versions are strengthenings of their coloristic counterparts and assert that every large subset of a "structure" must contain a "substructure". We remark that not all coloring results in Ramsey theory admit a density analogue—the classical Ramsey theorem [38] is a particularly striking example. However, when a density version is available, it provides a powerful structural information.

2.1. One of the most illuminating and well-known instances of the aforementioned phenomenon is the density version of the Hales–Jewett theorem, a result which is due to Furstenberg and Katznelson [24].

Density Hales–Jewett theorem. *For every integer* $k \geqslant 2$ *and every* $0 < \delta \leqslant 1$ *there exists a positive integer* N *with the following property. If* $n \geqslant N$*, then every subset* A *of* $[k]^n$ *with* $|A| \geqslant \delta k^n$ *contains a combinatorial line of* $[k]^n$*. The least positive integer with this property will be denoted by* $\mathrm{DHJ}(k, \delta)$*.*

The density Hales–Jewett theorem is a fundamental result of Ramsey theory. It has many strong results as consequences, most notably the famous Szemerédi theorem on arithmetic progressions [44] and its multidimensional version [22].

[2]We recall that if $w = (w_i)_{i=0}^{n-1}$ is a variable word over k of length n, then its *wildcard set* is defined to be the set $\big\{i \in \{0, \dots, n-1\} : w_i = v\big\}$.

[3]A tree T is *homogeneous* if it is uniquely rooted and there exists an integer $b \geqslant 2$, called the *branching number* of T, such that every $t \in T$ has exactly b immediate successors; e.g., every dyadic, or triadic tree is homogeneous.

Several different proofs of the density Hales–Jewett theorem are now known; see [3, 36, 47]. The most effective one is Polymath's proof [36] which gives the best known upper bounds for the numbers $\mathrm{DHJ}(k,\delta)$.

Yet another proof of the density Hales–Jewett theorem was discovered by the author, Kanellopoulos and Tyros in [17]. It grew out of the techniques developed in the course of obtaining a density version of the Carlson–Simpson theorem, a result which we will discuss in detail in §2.3. It appears that this is the simplest known proof of this deep result, while on the same time gives essentially the same upper bounds for the numbers $\mathrm{DHJ}(k,\delta)$ as in [36]. These upper bounds, however, are admittedly weak and have an Ackermann-type dependence with respect to k. It is one of the central open problems of Ramsey theory to decide whether these estimates can be significantly improved.

Problem 1. *Is it true that the numbers* $\mathrm{DHJ}(k,\delta)$ *are upper bounded by a primitive recursive function?*

2.2. The natural problem whether the Halpern–Läuchli theorem admits a density version was first asked by Laver in the late 1960s who actually conjectured that there is such a version. The conjecture was circulated among experts in the area and it was explicitly stated by Bicker and Voigt in [7]. It took slightly more than expected to decide this problem and the conjecture was finally settled affirmatively in [13].

Density Halpern–Läuchli theorem. *For every finite tuple* $(T_1,\dots,T_d)$ *of homogeneous trees and every subset* A *of the level product of* $(T_1,\dots,T_d)$ *satisfying*

$$\limsup_{n\to\infty}\frac{|A\cap\big(T_1(n)\times\cdots\times T_d(n)\big)|}{|T_1(n)\times\cdots\times T_d(n)|}>0 \tag{3}$$

there exist strong subtrees $(S_1,\dots,S_d)$ *of* $(T_1,\dots,T_d)$ *of infinite height and with a common level set such that their level product is a subset of* A.

We should point out that the assumption in the above result that the trees $T_1,\dots,T_d$ are homogeneous is not redundant. On the contrary, various examples given in [7] show that it is essentially optimal.

Just as many other infinite-dimensional results in Ramsey theory, the density Halpern–Läuchli theorem was expected to have a finite counterpart. Isolating, however, the proper finite analogue was a finer issue than anticipated. This was done in [15] where the following theorem was proved.

Density Halpern–Läuchli theorem (finite version). *For every integer* $d\geqslant 1$, *every* $b_1,\dots,b_d\in\mathbb{N}$ *with* $b_i\geqslant 2$ *for all* $i\in[d]$, *every integer* $m\geqslant 1$ *and every real* $0<\delta\leqslant 1$ *there exists a positive integer* N *with the following property. If* $(T_1,\dots,T_d)$ *are homogeneous trees such that the branching number of* T_i *is* b_i *for all* $i\in[d]$, L *is a finite subset of* $\mathbb{N}$ *of cardinality at least* N *and* D *is a subset of* *the level product of* $(T_1, \dots, T_d)$ *satisfying*

$$|D \cap \big(T_1(n) \times \cdots \times T_d(n)\big)| \geqslant \delta |T_1(n) \times \cdots \times T_d(n)| \tag{4}$$

for every $n \in L$*, then there exist strong subtrees* $(S_1, \dots, S_d)$ *of* $(T_1, \dots, T_d)$ *of height* m *and with a common level set such that their level product is contained in* D*. The least positive integer with this property will be denoted by* $\mathrm{UDHL}(b_1, \dots, b_d | m, \delta)$*.*

The main point here is that the result is independent of the position of the finite set L. We emphasize that this is a strong structural property that does not follow from the corresponding infinite version via standard compactness arguments. On the contrary, it can be used to derive the infinite version. The method of the reduction was discussed in detail in [16, §1.3]. It has already been applied in a different context, and should be investigated further and understood better.

We also remark that the proof in [15] is effective and yields explicit upper bounds for the numbers $\mathrm{UDHL}(b_1, \dots, b_d | m, \delta)$. However, just as in the case of the density Hales–Jewett theorem, these estimates are weak and have an Ackermann-type dependence with respect to the "dimension" d.

2.3. It is likely that the reader has already wondered whether the Carlson–Simpson theorem also has a density analogue. This problem was recognized by several groups of researchers as a key step in this line of research and was answered affirmatively, very recently, in [18].

Density Carlson–Simpson theorem. *For every integer* $k \geqslant 2$ *and every set* A *of words over* k *satisfying*

$$\limsup_{n \to \infty} \frac{|A \cap [k]^n|}{k^n} > 0 \tag{5}$$

there exist a word c *over* k *and a sequence* (w_n) *of left variable words over* k *such that the set*

$$\{c\} \cup \big\{ c^\frown w_0(a_0)^\frown \dots ^\frown w_n(a_n) : n \in \mathbb{N} \text{ and } a_0, \dots, a_n \in [k] \big\} \tag{6}$$

is contained in A*.*

We have already indicated that the proof of the density Carlson–Simpson theorem follows the strategy developed in [16]. In particular, the proof is based on a reduction of the infinite version to an appropriate finite one. This finite version, which represents the combinatorial core of the above result, is the content of the following theorem also proved in [18].

Density Carlson–Simpson theorem (finite version). *For every integer* $k \geqslant 2$*, every integer* $m \geqslant 1$ *and every* $0 < \delta \leqslant 1$ *there exists a positive integer* N *with the following property. If* L *is a finite subset of* $\mathbb{N}$ *of cardinality at least* N *and* A *is a set of words over* k *satisfying* $|A \cap [k]^n| \geqslant \delta k^n$ *for every* $n \in L$*, then there exist a* *word c over k and a finite sequence $(w_n)_{n=0}^{m-1}$ of left variable words over k such that the set*

$$\{c\} \cup \big\{c^\frown w_0(a_0)^\frown \dots ^\frown w_n(a_n) : n \in \{0, ..., m-1\} \text{ and } a_0, \dots, a_n \in [k]\big\} \qquad (7)$$

is contained in A. The least positive integer with this property will be denoted by $\mathrm{DCS}(k, m, \delta)$.

Again we emphasize that the main point here is that the result is independent of the position of the finite set L.

It is easily seen that the density Carlson–Simpson theorem implies both the density Hales–Jewett theorem and the density Halpern-Läuchli theorem. In fact we have finer quantitative information. Specifically, by [18, Proposition 11.13], it follows that

$$\mathrm{DHJ}(k, \delta) \leqslant \mathrm{DCS}(k, 1, \delta). \qquad (8)$$

On the other hand, using standard arguments (see, e.g., [9, 37]) we get that

$$\mathrm{UDHL}(b_1, \dots, b_d|m, \delta) \leqslant \mathrm{DCS}\Big(\prod_{i=1}^{d} b_i, m, \delta\Big). \qquad (9)$$

These remarks and the two metric relations isolated above indicate that the density Carlson–Simpson theorem is a statement which is centrally located in this part of Ramsey theory.

Also we notice that the argument in [18] yields explicit upper bounds for the numbers $\mathrm{DCS}(k, m, \delta)$. Unfortunately, these upper bounds are rather weak and shed no light on the behavior of the invariants $\mathrm{DHJ}(k, \delta)$ and $\mathrm{UDHL}(b_1, \dots, b_d|m, \delta)$. Nevertheless, it is natural to expect that there exist significantly stronger estimates.

Conjecture 2. *The numbers* $\mathrm{DCS}(k, m, \delta)$ *are upper bounded by a primitive recursive function.*

Notice, in particular, that an affirmative answer to Conjecture 2 will automatically settle in the affirmative Problem 1. While we strongly believe on the validity of Conjecture 2, we remark that such an achievement is certainly out of reach of current technology and is likely to require radical new ideas.

3. The probabilistic versions

The proofs of the results mentioned in §2 require a number of tools and a variety of techniques whose proper exposition is a lengthy and delicate task. We will thus narrow down our discussion to a particular part of the argument which is nevertheless central to the approach. In fact, this part of the argument is just an instance of yet another general phenomenon in Ramsey theory that concerns the structure of measurable events in probability spaces indexed by a *Ramsey space* [8].

The phenomenon is most transparently seen when the events are indexed by the natural numbers $\mathbb{N}$, an archetypical Ramsey space. Specifically, let (Ω, Σ, μ) be a probability space and assume that we are given a family $\{A_i : i \in \mathbb{N}\}$ of measurable events in (Ω, Σ, μ) satisfying $\mu(A_i) \geqslant \varepsilon > 0$ for every $i \in \mathbb{N}$. Using Ramsey's classical theorem [38] and elementary probabilistic estimates (see, e.g., [21]), it is easy to see that for every $0 < \theta < \varepsilon$ there exists an infinite subset L of $\mathbb{N}$ such that for every integer $n \geqslant 1$ and every subset F of L of cardinality n we have

$$\mu\Big(\bigcap_{i\in F} A_i\Big) \geqslant \theta^n. \tag{10}$$

In other words, the events in the family $\{A_i : i \in L\}$ are at least as correlated as if they were independent.

Now suppose that the events are not indexed by the natural numbers but are indexed by another Ramsey space $\mathcal{S}$. A natural problem, which is of combinatorial and analytical importance, is to decide whether the aforementioned result is valid in the new setting.

Problem 3. *Given a family $\{A_s : s \in \mathcal{S}\}$ of measurable events in a probability space (Ω, Σ, μ) indexed by a Ramsey space $\mathcal{S}$ and satisfying $\mu(A_s) \geqslant \varepsilon > 0$ for every $s \in \mathcal{S}$, is it possible to find a "substructure" $\mathcal{S}'$ of $\mathcal{S}$ such that the events in the family $\{A_s : s \in \mathcal{S}'\}$ are highly correlated? And if yes, then can we get explicit* (*and, hopefully, optimal*) *lower bounds for their joint probability?*

Of course, the notion of "substructure" will depend on the nature of the given index set $\mathcal{S}$. In all cases of interest, this problem is essentially equivalent to that of finding "copies" of given configurations inside dense sets of discrete structures. The equivalence between the two perspectives is discussed in detail in [18, §8.1] and is based on the "regularity method", a remarkable discovery of Szemerédi [45] asserting that dense sets of discrete structures are inherently pseudorandom.

3.1. A systematic study of Problem 3 was initiated in [14] where the case of a family of events indexed by a homogeneous tree was treated. Of course, the problem is also of interest when the index set is a tree belonging to a wider class. However the critical case is that of homogeneous trees—see [14, Appendix A]—and in this direction the following theorem was proved.

Theorem A. *Let T be a homogeneous tree with branching number b. Also let $\{A_t : t \in T\}$ be a family of measurable events in a probability space (Ω, Σ, μ) satisfying $\mu(A_t) \geqslant \varepsilon > 0$ for every $t \in T$. Then for every $0 < \theta < \varepsilon$ there exists a strong subtree S of T of infinite height such that for every integer $n \geqslant 1$ and every subset F of S of cardinality n we have*

$$\mu\Big(\bigcap_{t\in F} A_t\Big) \geqslant \theta^{q(b,n)} \tag{11}$$

where

$$q(b,n) = \frac{(2^b-1)^{2n-1}-1}{2^b-2}. \tag{12}$$

Observe that the integer $q(b,n)$ obtained above depends exponentially on n. Nevertheless, the estimate in (12) is fairly "civilized" and is arguably one of the best bounds available in this part of Ramsey theory. A basic ingredient of the proof of Theorem A—an ingredient which is partly responsible for the effectiveness of the argument—is an appropriate generalization of the notion of a "Shelah line", a fundamental tool in Ramsey theory introduced by Shelah in his work [40] on the van der Waerden and the Hales–Jewett numbers.

3.2. In [16] the above analysis was extended to the higher-dimensional setting, namely when we deal with events indexed by the level product of a vector homogeneous tree. Recall that a *vector homogeneous tree* $\mathbf{T}$ is a finite sequence $(T_1,\dots,T_d)$ of homogeneous trees and its *level product* $\otimes\mathbf{T}$ is the subset of the Cartesian product $T_1\times\cdots\times T_d$ consisting of all finite sequences $(t_1,\dots,t_d)$ of nodes having common length. The corresponding notion of "substructure" is that of a *vector strong subtree*. Specifically, a vector strong subtree of a vector homogeneous tree $\mathbf{T}=(T_1,\dots,T_d)$ is just a finite sequence $\mathbf{S}=(S_1,\dots,S_d)$ of strong subtrees of $(T_1,\dots,T_d)$ having a common height and a common level set.

The following result is the higher-dimensional analogue of Theorem A and was proved in [16].

Theorem B. *For every integer $d\geqslant 1$, every $b_1,\dots,b_d\in\mathbb{N}$ with $b_i\geqslant 2$ for all $i\in[d]$, every integer $n\geqslant 1$ and every $0<\varepsilon\leqslant 1$ there exists a strictly positive constant $c(b_1,\dots,b_d|n,\varepsilon)$ with the following property. If $\mathbf{T}=(T_1,\dots,T_d)$ is a vector homogeneous tree such that the branching number of T_i is b_i for all $i\in[d]$ and if $\{A_{\mathbf{t}}:\mathbf{t}\in\otimes\mathbf{T}\}$ is a family of measurable events in a probability space (Ω,Σ,μ) satisfying $\mu(A_{\mathbf{t}})\geqslant\varepsilon$ for every $\mathbf{t}\in\otimes\mathbf{T}$, then there exists a vector strong subtree $\mathbf{S}=(S_1,\dots,S_d)$ of $\mathbf{T}$ of infinite height such that for every integer $n\geqslant 1$ and every subset F of the level product $\otimes\mathbf{S}$ of $\mathbf{S}$ of cardinality n we have*

$$\mu\Big(\bigcap_{\mathbf{t}\in F}A_{\mathbf{t}}\Big)\geqslant c(b_1,\dots,b_d|n,\varepsilon). \tag{13}$$

The proof of Theorem B is effective and yields explicit estimates for the constants $c(b_1,\dots,b_d|n,\varepsilon)$. These estimates, however, rely heavily on the bounds obtained in [15] and are, therefore, rather poor. It is a challenging open problem to obtain "civilized" bounds for these constants, a problem which is closely related to that of improving the upper bounds for the numbers $\mathrm{UDHL}(b_1,\dots,b_d|m,\delta)$.

3.4. We conclude this review with a discussion on the results in [19] dealing with a family of measurable events indexed by words. In this context the appropriate notion of "substructure" is that of a *Carlson–Simpson tree* [9, 18]. Recall that a Carlson–Simpson tree of $[k]^{<\mathbb{N}}$ of dimension $m\geqslant 1$ is a subset of $[k]^{<\mathbb{N}}$ of the form

$$\{c\}\cup\big\{c^\frown w_0(a_0)^\frown\dots{}^\frown w_n(a_n):n\in\{0,\dots,m-1\}\text{ and }a_0,\dots,a_n\in[k]\big\} \tag{14}$$

where c is a word over k and $(w_n)_{n=0}^{m-1}$ is a nonempty finite sequence of left variable words over k.

The following theorem was proved in [19] and is a strong "probabilistic" version of the density Carlson–Simpson theorem.

Theorem C. *For every integer $k \geqslant 2$, every $0 < \varepsilon \leqslant 1$ and every integer $n \geqslant 1$ there exists a strictly positive constant $\theta(k, \varepsilon, n)$ with the following property. If m is a given positive integer, then there exists a positive integer $\mathrm{Cor}(k, \varepsilon, m)$ such that for every Carlson–Simpson tree T of $[k]^{<\mathbb{N}}$ of dimension at least $\mathrm{Cor}(k, \varepsilon, m)$ and every family $\{A_t : t \in T\}$ of measurable events in a probability space (Ω, Σ, μ) satisfying $\mu(A_t) \geqslant \varepsilon$ for every $t \in T$, there exists a Carlson–Simpson tree S of dimension m with $S \subseteq T$ and such that for every nonempty $F \subseteq S$ we have*

$$\mu\Big(\bigcap_{t \in F} A_t \Big) \geqslant \theta(k, \varepsilon, |F|). \tag{15}$$

The invariants of main interest in Theorem C are, of course, the constants $\theta(k, \varepsilon, n)$. The argument in [19] does yield quantitative information for these invariants but, unsurprisingly, this information is weak. However, more important is the fact that for the proof of Theorem C a number of tools needed to be developed, including a refinement of a partition result due to Furstenberg and Katznelson [23]. This refinement is of independent interest and forms the basis for a complete classification of those classes of subsets of Carlson–Simpson trees which are partition regular. The details of this classification will appear in [12].

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